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A novel mass-spring model with triangle meshes for modeling membranes

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Abstract. Mass-Spring Models (MSM) are wildly used to simulate the mechanical behavior of deformable bodies because of their simplicity and computational efficiency. However, the major difficulty in using MSMs lies on the spring stiffness estimation since the parameters are typically determined through trial-and-error. Only a few contributions are made to establish the analytical relation between the spring parameters and elastic material properties. In this paper, a novel MSM with three additional nodes (MSMTN) for the membrane is presented based on continuum mechanics of deformable objects. The spring parameters of the MSMTN in triangle elements have analytical expressions suited to arbitrary isotropic material parameters and element shapes without any limitation. The proposed MSMTN can achieve same element stiffness matrix with the one in finite element model (FEM) for triangle membrane element, thereby the MSMTN can reach the similar computational accuracy with FEM. This feature of MSMTN makes up the weakness in computation accuracy of classical MSMs effectively. Numerical simulations of tensile membranes show that the MSMTN and improved deformation algorithm have satisfactory accuracy and good generality.

Key words. Mass-Spring Model, finite element model, spring parameter, membrane, additional node..

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1. Introduction

Fast and accurate deformation simulation of the flexible membrane is an important task in different fields such as computer graphics and membrane structural analysis. The Finite Element Method (FEM) and the Mass-Spring Model (MSM) are two main approaches used to describe the mechanical behavior of flexible membrane. Based on the continuum constitutive relationships of elastic bodies, FEM has attractive computational efficiency for linear problems, but the computation cost increases considerably when nonlinear FEM is used to solve the problems with large deformation or sophisticate material models. Comparing with FEM, MSMs are essentially discrete models which consist of discrete springs and lumped masses with different mesh topology. The deformation of the flexible body is described as the position changes of mass points, and the motion equations of each particle are formulated by Newtonian second law. MSMs are usually believed to be simple and efficient, but not as accurate as FEM. The drawback of MSM in computational accuracy comes mainly from that the formulas of spring parameters do not include the elastic material constitutive laws.

Determination of the parameters like spring stiffness in MSMs still remains a great challenge [1]. The derivation methods of parameters can be classified as data-driven methods and analytical derivation methods loosely. Data-driven approaches can obtain the optimized spring stiffness or mesh topology by fitting the deformation of MSMs to the reference data. However, the parameters optimization usually requires expensive computation, and will be repeated when any alteration about material, mesh or the optimal algorithm is made. In contrast to the data-driven methods, analytical methods try to derive analytical expressions of the parameters in MSMs by using continuum mechanics and finite element theory. The natural discrete feature of MSMs makes the analytical derivation is difficult, so there are only a few contributions in this research area. Gelder [2] concluded that triangulated 'spring mesh model cannot exactly simulate the membrane model, in sense of having the same equilibria.', and then, an approximate expression was suggested.

In this paper, a new MSM with three additional nodes (MSMTN) is proposed to simulate the thin membrane. MSMTN has the same equivalent element stiffness matrix with linear FEM and the spring coefficients in MSMTN also have the explicit analytical expressions. Moreover, the spring coefficient expressions are more concise and suitable for the arbitrary triangle shape and isotropic materials without any limitation. The corresponding space deformation algorithm for the MSMTN is improved to compute the displacement of additional nodes. Numerical cases show that the MSMTN and improved deformation algorithm have satisfactory accuracy and good generality.

2. MSMTN and identification of spring parameters

In finite element methods, the element stiffness matrix can represent the physical relationship between nodal deformations and loads, and is determined by element shape and material properties. Here, we will determine the spring parameters of

MSMTN by comparing with FEM for isotropic membrane. Three additional springs and additional nodes are introduced as independent controlled parameters, so that the stiffness matrix of the MSM is equal to the stiffness matrix of the FEM. Consider the elastic membranes as a uniform, orthotropic and linear elastic material, and the assumption of plane stress is used in this study.

2.1. Stiffness Matrix of a Triangle element in FEM

3-node Triangle element is a common 2D element to simulate the thin membrane. As a constant strain and stress element, the stiffness matrix of triangle element [9] can be expressed analytically as

$$K_{fem}^{e} = \begin{bmatrix} K_{i,i} & K_{i,j} & K_{i,k} \\ K_{j,i} & K_{j,j} & K_{j,k} \\ K_{k,i} & K_{k,j} & K_{k,k} \end{bmatrix} \tag{1}$$

$$K_{msm}^{e} d_{msm}^{e} = \begin{bmatrix} K_{mm}^{e} & K_{ms}^{e} \\ K_{sm}^{e} & K_{ss}^{e} \end{bmatrix} \begin{Bmatrix} d_{m}^{e} \\ d_{s}^{e} \end{Bmatrix} = \begin{Bmatrix} F_{m}^{e} \\ F_{s}^{e} \end{Bmatrix} \tag{2}$$

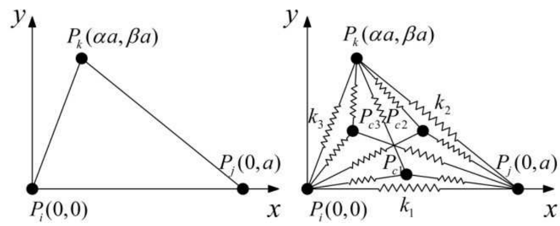

Fig. 1. Triangle element in (a) FEM (b) MSMTN

2.2. Stiffness Matrix of a Triangle element in MSMTN

The MSM triangle element with three additional nodes is illustrated in Fig (1b). The vertexes P_i, P_j and P_k are taken as master nodes, and the additional nodes P_{c1} , P_{c2} and P_{c3} are defined as the slave nodes. Three vertexes and the center of gravity of the triangle can form three small triangles, and the initial location of additional nodes are set at the centers of the gravity of three small triangles, so the initial coordinates of additional nodes are

$$P_{c1}\left(\frac{4+\alpha}{9}a, \frac{\beta}{9}a\right)$$

,

$$P_{c2}\left(\frac{4+4\alpha}{9}a, \frac{4\beta}{9}a\right)$$

and

$$P_{c3}\left(\frac{1+4\alpha}{9}a, \frac{4\beta}{9}a\right)$$

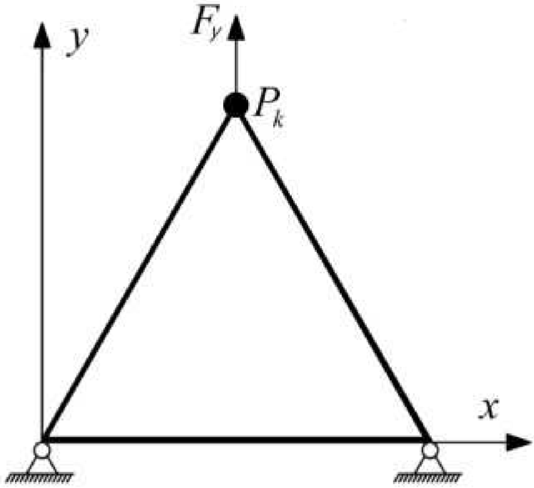

Fig. 2. Single triangle element. (a)(b)

. The stiffness parameters of the springs connected with master nodes are k_1 , k_2 and k_3 , and k_{c1} is defined as the stiffness of the spring connected master nodes to additional node P_{c1} . The spring stiffness k_{c2} and k_{c3} are defined similarly.

In proposed MSMTN, three assumption about additional node are adopted (i) the mass and damp of the additional nodes are neglected (ii) the additional nodes do not undertake the external loading directly (iii) the additional node is always located in the plane of the triangle element. Here, only the displacements of the master nodes in MSMTN are required to accord with the deformation of actual structures. The main function of the additional nodes and springs is adjusting the relationship between the force and deformation to simulate actual membrane. Therefore, MSMTN provide three extra independent variables k_{c1}, k_{c2} and k_{c3}, which make it possible to match the stiffness matrix of MSM with the corresponding matrix of FEM accurately.

where the subscripts m and s denote the degree of freedoms (dofs) of the master and additional nodes, respectively. d_m^e and d_s^e represent the displacement of the master and additional nodes, F_m^e and F_s^e are the external force on the master and additional nodes, respectively. Using the assumption (ii) about additional nodes, the load in additional nodes $F_s^e = 0$, so the relation between the external force and the displacement of the master nodes can be obtained from Eq.(2).

$$F_m^e = K_{msmtn}^e = (K_{mm}^e - K_{ms}^e (K_{ss}^e)^{-1} K_{sm}^e) d_m^e \tag{3}$$

$$J = \left\| K_{msmtn}^e - K_{fem}^e \right\|_2^2 \tag{4}$$

Apparently, the equivalent stiffness matrix K_{msmtn}^e is the function of element shape parameters α, β and spring parameters k_1 , k_2 ,k_3 , k_{c1}, $k_{c2} = \eta k_c$, $k_{c3} = \theta k_c$, where η and θ are dimensionless variables.

2.3. Derivation of Spring Parameters in MSMTN

Define the Euclidean distance of the element stiffness matrices between FEM and MSMTN as

where K^e_{msmtn} is the function of $\alpha, \beta, k_1, k_2, k_3, k_c, \eta, \theta$, and K^e_{fem} is the function of α, β, E, ν, t, where t is the thickness of thin membrane, E is the Young's modulus and ν is the Poisson's ratio of the material. Here the objective function J is a continuous differentiable function.

The objective function J has the minimum value $J_{\min}$=0 only when

$$K^e_{msmtn} = K^e_{fem}$$

. If the minimum value$J_{\min}$ =0 exists, the following equations should be satisfied

$$\partial J/\partial k_l = 0, \quad l = 1, 2, 3, c \tag{5}$$

The spring parameters k_1, k_2 , k_3 and k_c can be obtained by solving the Eq.(5), and then the corresponding results are substituted into $J_{\min}$ =0, the simplified equation is

$$J\left(\alpha, \beta, \nu, \eta, \theta\right) = 0 \tag{6}$$

The simplified equation (6) is a quadratic equation of the variable η, so two different solutions of η can be obtained as

$$\eta = \frac{f_1(\alpha,\beta,\nu,\theta) \pm \sqrt{g(\alpha,\beta,\nu,\theta)}}{f_2(\alpha,\beta,\nu,\theta)} \tag{7}$$

where the expressions of the function f_1, f_2 and g are too long to show in detail. Because of the uniqueness of the additional spring parameter, let $g\left(\alpha, \beta, \nu, \theta\right)$=0, and then the analytical expression of θ can be deduced. Using the expressions of the dimensionless variables η and θ , the spring parameters in MSMTN can be deduced and simplified as

$$k_1 = 9(-12\alpha^3 + 6\alpha^4 + 12\alpha^2(-2 + \beta^2) - 6\alpha(-5 + 2\beta^2) + \beta^2(-5 + 6\beta^2 - 9\nu))k_0 \tag{8}$$

For equilateral triangle element ($\alpha = \frac{1}{2}$,$\beta = \frac{\sqrt{3}}{2}$) with the Poisson's ratio 1 / 3 , the Eqs. (8)can be simplified as

$$k_1 = k_2 = k_3 = \frac{\sqrt{3}}{4} Et, k_{c1} = k_{c2} = k_{c3} = 0.$$

The deduced spring parameters are same completely with Lloyd's formula [3].

The solutions (8)can ensure $J_{\min}$ =0, thereby the stiffness matrix in MSMTN is exactly the same as that in FEM. The deduced analytical expressions of the spring parameters in MSMTN are suitable for arbitrary triangle element and materials properties without any limitation.

3. Deformation algorithm of the MSMTN

The deformation algorithm of the MSMTN is different with classical MSMs since the introduction of additional springs and nodes. The displacement of master nodes and additional nodes should be calculated respectively because of different characteristic. According to the assumption about additional nodes in Section 2.2, the dynamic equations about master and additional nodes in the MSMTN can be expressed respectively as

$$M_{mm}\ddot{d}_m + C_{mm}\dot{d}_m + F_{spring} = F_m \tag{9}$$

$$K_{sm}d_m + K_{ss}d_s = F_s = 0 \tag{10}$$

whered_m and d_s are the displacement vector of all master nodes and additional nodes respectively, F_{spring} is the spring force acted on the master nodes and can be calculated by the deformation of springs in MSMTN.

3.1. Deformation of the Master Nodes

The deformation of master nodes in MSMTN can be computed by the same algorithms with classical MSMs. In this paper, the time step integral method proposed by Verlet is used to solve the Eq. (9)only for its simplicity, and it can be replaced by other numerical integral algorithms. Using the initial value or the results of the previous time step, new acceleration, velocity and displacement of the master nodes at the next time step can be calculated.

3.2. Deformation of the Additional Nodes

Because an additional node only connects with three master nodes in the same element, the deformation of additional nodes can be solved by using the Eq. (10) in planar problems

$$d_s^e = \left(K_{ss}^e\right)^{-1}\left(-K_{sm}^e d_m^e\right) = \left(K_{ss}^e\right)^{-1} F_{sm}^e \tag{11}$$

However, three additional springs located in the plane of the triangle element cannot provide the out-of-plane stiffness, so the rigid displacement of the triangle element leads to a singular stiffness matrix K_{ss}^e. In order to eliminate the singularity of K_{ss}^e, the in-plane and out-of-plane displacement of additional nodes are calculated separately. The in-plane component is caused by the elastic deformation of the triangle MSM element, while the out-of-plane component of the additional nodes $\Delta_c = \{d_{cx}, d_{cy}, d_{cz}\}^T$ is decided by the rigid displacement of the master nodes $\Delta_m = \{\Delta_i, \Delta_j, \Delta_k\}^T$:

$$\Delta_c = \chi \cdot \Delta_m \tag{12}$$

where $\chi = \{\chi_i, \chi_j, \chi_k\}$,

$$\chi_i = (n_{ijk} \cdot n_{jkc})\frac{S_{\Delta jkc}}{S_{\Delta ijk}}, \chi_j = (n_{ijk} \cdot n_{kic})\frac{S_{\Delta kic}}{S_{\Delta ijk}}, \chi_k = (n_{ijk} \cdot n_{ijc})\frac{S_{\Delta ijc}}{S_{\Delta ijk}}.$$

S_Δ is the area of corresponding triangle, and the unit vectors are defined as

$$n_{ijk} = \frac{\overrightarrow{P_iP_j} \times \overrightarrow{P_iP_k}}{\left|\overrightarrow{P_iP_j} \times \overrightarrow{P_iP_k}\right|}, n_{jkc} = \frac{\overrightarrow{P_jP_k} \times \overrightarrow{P_jP_c}}{\left|\overrightarrow{P_jP_k} \times \overrightarrow{P_jP_c}\right|}, n_{kic} = \frac{\overrightarrow{P_kP_i} \times \overrightarrow{P_kP_c}}{\left|\overrightarrow{P_kP_i} \times \overrightarrow{P_kP_c}\right|}, n_{ijc} = \frac{\overrightarrow{P_iP_j} \times \overrightarrow{P_iP_c}}{\left|\overrightarrow{P_iP_j} \times \overrightarrow{P_iP_c}\right|}.$$

Then, we can compute the complete displacement of additional nodes by solving the simultaneous equations consisted of two equations in Eq. (11)and one equation in Eq.(12).

The deformation algorithm of MSMTN has more computational cost than classical MSMs because of the displacement calculation of the additional nodes. It is lucky that the displacement of each additional node can be computed individually by solving the equations with the rank 3. Furthermore, ideal calculation precision of MSMTN allows lower mesh resolutions which also contribute to reduce the computational cost. Thereby, high computational efficiency still can be maintained in new MSMTN in dynamical simulation.

4. Experiments and resultsw

In this section, two numerical examples about tensile membranes are presented to display the performance of MSMTN. Without loss of generality, dimensionless variables are used in the cases.

4.1. Single Triangle Element

In classical MSMs, negative stiffness springs are rejected due to the potential difficulty in calculation. However, the parameters of both structural and additional springs are possible to be negative in MSMTN, so it is important to ensure the correctness of the deformation calculation in the MSMTN with negative stiffness springs.

In this case, two different triangle membrane elements with the thickness t=0.001 are selected, as shown in Fig. (2a) and (2b), and corresponding material properties and loads are listed in Table 1. The spring parameters in MSMTN can be calculated by Eq.(8)according to the triangular shape and material properties, the results are listed in Table **2**.

The bottom side of triangle elements are fixed, and the different tensile forces F_x and F_y are applied to the node P_k. The displacement of node P_k obtained by MSMTN and FEM are listed in Table **3**, Table **4** and Table **5**, respectively. Here, the results of FEM are computed by ANSYS software. The results show the MSMTN has similar computational accuracy with FEM.

Table 1. Shape parameters of triangle membrane elements, material properties and load

Element	α	β	E	ν
Fig. (2a)	0.5	$\sqrt{3}/2$	100000	0.25
Fig. (2b)	0.7	0.6	100000	0.40

Table 2. Spring parameters in MSMTN

Element	k_1	k_2	k_3	k_{c1}	k_{c2}
Fig. (2a)	146	146	146	-12.8	-12.8
Fig. (2b)	460	74.0	69.6	-339	168

Table 3. Displacement along y-direction (UY) of vertex P_k of the element in Fig. 2(a)

F_y	**0.1**	**0.5**	**1**	**2**	**3**	**4**
UY×10^{-2} (MSMTN)	0.21162	1.0583	2.1174	4.2391	6.3664	8.4999
UY×10^{-2} (FEM)	0.21163	1.0582	2.1163	4.2326	6.3489	8.4652
Relative error	0.00%	0.02%	0.05%	0.15%	0.28%	0.41%

Table 4. Displacement (UX and UY) of vertex P_k of the element in Fig. 2(b)

$F_x = F_y$	**0.01**	**0.05**	**0.1**	**0.2**
UY×10^{-3}(MSMTN)	0.33598	1.6799	3.3594	6.7179
UY×10^{-3}(FEM)	0.33600	1.6800	3.3600	6.7200
Relative error	0.005%	0.007%	0.016%	0.032%
UX×10^{-3} (MSMTN)	0.10082	0.50436	1.0094	2.0218
UX×10^{-3} (FEM)	0.10080	0.50400	1.0080	2.0160
Relative error	-0.016%	-0.071%	-0.144%	-0.286%

4.2. Square Membrane

A square membrane shown in Fig. (**3**) with dimension 1×1 and thickness 0.001 is the tested patch. The geometry is discretized with 200 elements in MSMTN. Four sides of square membrane are all fixed, and a tensile force $F = 0.1$ which is perpendicular to the plane of the membrane is applied on the central node of the square. The material properties are set as E=80000 and ν=0.4.

Fig.4 shows the deformation profile of the square membrane calculated by MSMTN. The super-elasticity (overstretch) in the zone near the concentrated load is a common problem in classical MSM, but it is observed that MSMTN can avoid this fault effectively. The displacement along z-direction of the central node is 0.0619271 in the MSMTN, and the result calculated by nonlinear FEM is 0.062164. The relative error is only 0.38%. The results indicate the MSMTN and deformation algorithm can solve the large deformation problems efficiently and accurately.

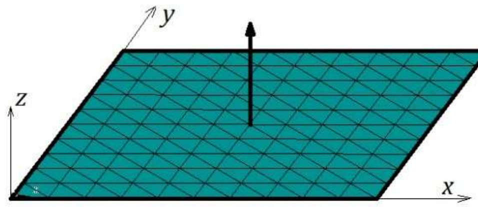

Fig. 3. Square membrane

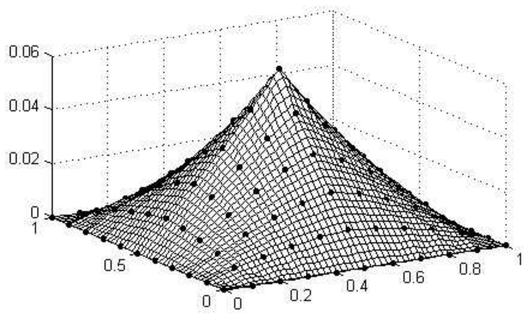

Fig. 4. Deformation profile of the square membrane

5. Conclusion

The study of vibration of plates of variable thickness is very important in a wide variety of applications in industry and engineering. In the present analysis desired frequencies of non-homogeneous trapezoidal plate whose thickness varies bilinearly and density varies parabolically in one direction has been studied through the Rayleigh–Ritz technique. The results corresponding to the C-S-C-S boundary condition are computed for some selected domain to obtain better understanding of the methodology. Effect of plate's parameters such as taper constants, thermal gradient, aspect ratio and non-homogeneity constant has also been considered for first two modes of vibration. The tables illustrate that the frequency parameter increases with the increase of taper constants and frequency parameter decreases with the increase of thermal gradient, aspect ratio and non-homogeneity constant. Comparison of present work has been made with known results and found to be much closed. The material should be selected such that the total cost should be minimum and within specified limits.

CONFLICT OF INTEREST

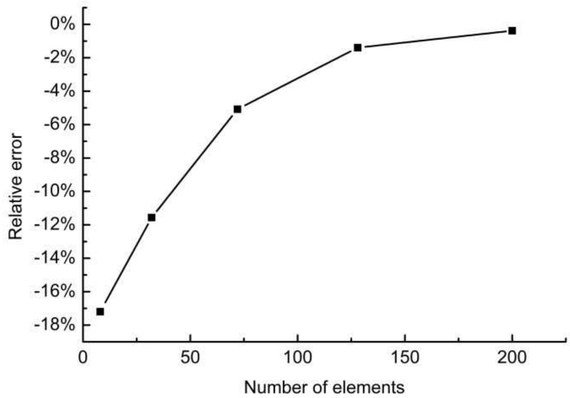

Fig. 5. The convergence of relative error

The author confirms that this article content has no conflict of interest.

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